Descriptive Set Theory
Lecture 7

Perfect Polish spaces. A top. space X is called *perfect* if it doesn't have any isolated points. A point $x \in X$ is called *isolated* if $\{x\}$ is open. A subset $Y \subseteq X$ is called *perfect* if it is closed and perfect.

Caution. $\mathbb{Q}$ is a perfect top. space but it's not a perfect subset of $\mathbb{R}$.

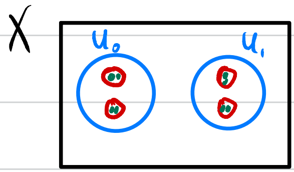
Examples of perfect Polish. $\mathbb{R}$, $[0,1]^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$, $[0,1]$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, $C([0,1])$ - continuous funct. on $[0,1]$, ℓ^p, L^p.

Counterexamples. $[0,1] \cup \{2\}$.

Perfect set theorem (Cantor). Any nonempty perfect Polish space X contains a homeomorphic copy of the Cantor space $2^{\mathbb{N}}$. In particular, X has cardinality continuum.

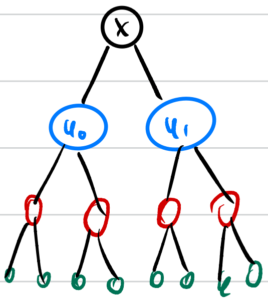
Proof. We build a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}}$ of vanishing diam. (wrt a complete compatible metric on X) s.t.
(i) $\overline{U_{s^\frown i}} \subseteq U_s$ (ii) $U_s \neq \emptyset$ open. This would guarantee that

the induced map f has domain $= 2^{\mathbb{N}}$ and it is a continuous injection (hence automatically an embedding).



Let $U_\emptyset := X$. Because $U_\emptyset$ is perfect, it has ≥ 2 points so Hausdorffness gives disjoint nonempty open U_0, U_1. For each U_s, $s \in 2^{<\mathbb{N}}$, that's already been defined, do the following:

$U_s \neq \emptyset \Rightarrow$ has at least two points $\Rightarrow$ contains two disjoint open balls of diam. $\leq 2^{-|s|}$ and such that their closures are contained in U_s. □



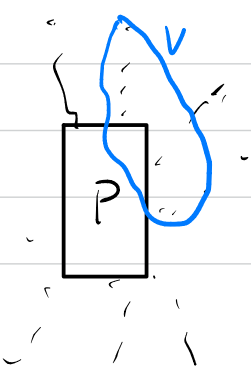
Theorem (Cantor–Bendixson). Every Polish space X can be uniquely written as a disjoint union $P \sqcup U$, where U is a ctbl open set and P is perfect. This P is called the *perfect core* of X.

Cor. Every Polish space has the *perfect set property*, i.e. it's either ctbl or contains a homeo. copy of $2^{\mathbb{N}}$ (hence has cardinality continuum).

Proof of uniqueness. Suppose $X = P_1 \sqcup U_1 = P_2 \sqcup U_2$, where the U_i are ctbl open and the P_i are perfect. It's enough to show that $P_1 \cap U_2 = \emptyset$. Suppose otherwise. Open subsets of perfect spaces are themselves perfect in the rel. top. and open subsets of Polish spaces are Polish. Thus, $P_1 \cap U_2$ is a nonempty perfect Polish space. Thus, $P_1 \cap U_2$ is unctbl, contradicting U_2 being ctbl. □

We give two proofs of the existence, in both of which we will be removing "small" open sets and eventually arriving at the perfect core. The proofs differ by which open sets are considered small: ctbl or finite.

Proof 1 of existence. Fix a ctbl basis (V_n) and let $\mathcal{V}$ be the collection of all basic open sets that are ctbl. Put $U := \bigcup \mathcal{V}$ and $P := X \setminus U$.



U is a ctbl union of ctbl open sets so it's ctbl and open. It remains to show that P is perfect. Fix a relatively open set $\emptyset \neq V \cap P$, where $V \subseteq X$ is open. Then $V \cap P$ has to be unctbl because otherwise V would be $(V \cap P) \sqcup (V \cap U)$ which is ctbl, but have removed such sets, i.e. $V \cap P \neq \emptyset$. Thus, $V \cap P$ has at least 2 elements. $\square$

For Proof 2, we need the notion of Cantor-Bendixson derivative: for a top. space X, let

$$X' := X \setminus \{x \in X : x \text{ is isolated}\},$$

and call it the Cantor-Bendixson derivative of X.

Let

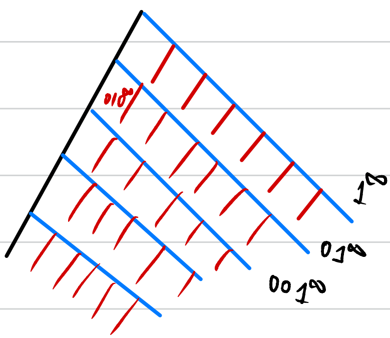
$$X^{(0)} := X,$$

$$X^{(\alpha+1)} := \left(X^{(\alpha)}\right)' \quad \text{for any ordinal } \alpha,$$

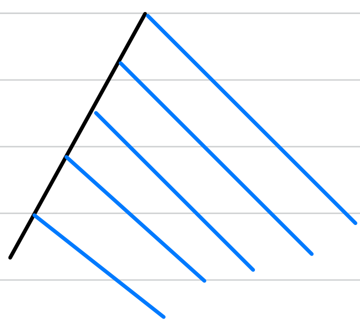
$$X^{\lambda} := \bigcap_{\alpha<\lambda} X^{(\alpha)} \quad \text{for a limit ordinal } \lambda.$$

Let X be the following subset of $2^{\mathbb{N}}$:

X



X'



X''

$X''' = \emptyset$

Note that when X is 2^{nd} ctbl, $X \setminus X'$ is ctbl because it is a disjoint union of 1-element open sets, which hence must be basic for any basis.

Proof 2 of existence. Consider the Cantor-Bendixson derivation $(X^{(\alpha)})_{\alpha<\omega_1}$. Note that this is a decreasing ordinal-indexed sequence of closed sets in a 2^{nd}-ctbl space, so it stabilizes at a ctbl ordinal $\alpha < \omega_1$, i.e. $X^{(\alpha+1)} = X^{(\alpha)}$. Thus, $X^{(\alpha)}$ is perfect and $X \setminus X^{(\alpha)} = \bigcup_{\beta<\alpha} X^{(\beta)} \setminus X^{(\beta+1)}$ and $X^{(\beta)} \setminus X^{(\beta+1)} = X^{(\beta)} \setminus (X^{\beta})'$ is ctbl, so $X \setminus X^{(\alpha)}$ is ctbl open. $\square$

Let the least ordinal α s.t. $X^{(\alpha)} = X^{(\alpha+1)}$ be called the **Cantor-Bendixson rank** of X and denoted by $|X|_{CB} := \alpha$. We denote the perfect core of X by $X^{|X|_{CB}}$ or X^{∞}.

In the example above, the Cantor-Bendixson rank is 3.

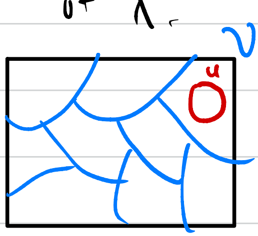
0-dimensional Polish spaces. A top. space is called **0-dimensional** if it admits a basis of clopen sets, e.g. $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$. These spaces are very disconnected, in fact, 0-dim. Hausdorff spaces are **totally**

disconnected, i.e. any distinct points x, y can be separated by disjoint clopen sets $U \ni x$ and $V \ni y$. The converse fails even for Polish spaces: Let X be the subset of $\ell^1(\mathbb{N}) :=$ sequences of reals that are absolutely summable, consisting of all points with irrational entries. This is a totally discon. G_δ subset of the Polish space $\ell^1(\mathbb{N})$, hence is itself Polish. But it is shown that this is not a 0-dimensional space (it is 1-dim).

0-dim Polish spaces admit a ctbl basis and also a basis consisting of clopen sets. Can we get a ctbl basis consisting of clopen sets? Yes.

Lemma. 2nd ctbl top. spaces are Lindelöf, i.e. every open cover admits a ctbl subcover.

Proof. Let $\mathcal{U}$ be a ctbl basis and let $\mathcal{V}$ be an open cover of X.



Let $\mathcal{U}' := \{U \in \mathcal{U} : \exists V \in \mathcal{V} \text{ containing } U\}$. This is still ctbl and note that $\mathcal{U}'$ covers X. For each $U \in \mathcal{U}'$ choose $V_U \in \mathcal{V}$ s.t. $V \supseteq U$. Then $\mathcal{V}' := \{V_U : U \in \mathcal{U}'\}$ is a ctbl subcover of $\mathcal{V}$. $\square$

Prop. For any 2^{nd} ctbl space X, any basis $\mathcal{B}$ admits a ctbl subcollection $\mathcal{B}' \subseteq \mathcal{B}$ that is still a basis.

Proof. Let (U_n) be a ctbl basis and let $\mathcal{B}$ be a basis. Every U_n is a union of sets in $\mathcal{B}$, i.e. admits a cover by sets in $\mathcal{B}$. But U_n itself is a 2^{nd} ctbl space so in fact U_n admits a ctbl cover by sets in $\mathcal{B}$. Putting all these covers together for all U_n, we obtain a still ctbl subset $\mathcal{B}' \subseteq \mathcal{B}$ s.t. $\forall n$, U_n is a union of sets in $\mathcal{B}'$, hence $\mathcal{B}'$ is a basis. $\square$